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Research Article

Star metric dimension of complete, bipartite, complete bipartite and fan graphs

Reni Umilasari[1✉], Ilham Saifudin[1] & Isnawati Lujeng Lestari[2]

[1] Department of Informatics Engineering, Universitas Muhammadiyah Jember, Jawa Timur, Indonesia, 68121
[2] Department of Mathematics Education, Institut Teknologi dan Sains Nahdlatul Ulama Pasuruan, Jawa Timur, Indonesia, 67171
✉Corresponding Author: reni.umilasari@unmuhjember.ac.id | Phone Number [+6285233372027]

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ABSTRACT

One of the topics in graph theory that is interesting and developed continuously is metric dimension. It has some new variation concepts, such as star metric dimension. The order set of $Z = \{z_1, z_2, \dots, z_n\} \subseteq V(G)$ called star resolving set of connected graph G if Z is Star Graph and for every vertex in G has different representation to the set Z. The representation is expressed as the distance $d(u,z)$, it is the shortest path from vertex u to z for every $u, z \in V(G)$. Star basis of a graph is the smallest cardinality of star resolving set. The number of vertex in star basis is called star metric dimension of G which denoted by $Sdim(G)$. The purpose of this article is to determine the characteristic of star metric dimension and the value of star metric dimension of some classes of graphs. The method which is used in this study is library research. Some of the results of this research are complete graph has $Sdim(K_n) = n - 1$, for $n \geq 3$, bipartite graph $K_{2,n}$ has $Sdim\left(K_{2,n}\right) = n$, for $n \geq 3$. Besides complete bipartite graph hasn't star metric dimension or for $m, n \geq 3$ or it can said that $Sdim\left(K_{m,n}\right) = 0$. Another graph, that is Fan graph has $Sdim\left(F_n\right) = 2$ for $2 \leq n \leq 5$ and for $n \geq 6$ $Sdim\left(F_n\right) = \left\lceil \frac{2n+3}{5} \right\rceil$.

Keywords: Metric Dimension; Star Basis; Star Metric Dimension; Star Resolving Set; Complete Bipartite

1. INTRODUCTION

The graph theory was first introduced by Leonhard Euler in 1736 with the case of the Konisberg Bridge. The problem that arises is known as the Konisberg bridge mystery, namely how to cross the four landmasses by crossing the seven bridges exactly once. To analyze the problem of the Konisberg bridge, Euler presents the land as a vertex while the bridge is presented as an edge. Euler argued that it was impossible for a person to cross each of the seven bridges exactly once. The proof is suspected to be the beginning of the emergence of the concept of graph (Kuziak et al., 2017). One of some topics in graph theory that is interesting and developed continuously is metric dimension. The theory of metric dimension was firstly proposed by Harary and Melter in 1976. The concept can be utilized to distuingish each vertex in a connected graph G by determining its representation with respect to subset of vertex set of G (Saputro et al., 2017). To get some new well known graph, some researcher introduced operation of some classes of graphs. Then, metric dimension of corona product graphs was been researched by (Yero et al., 2011).

Another operation is strong product graphs. Fernau H and Rodríguez-Velázquez J investigate the adjacency metric dimension of corona and strong product graphs. The results of their research are about the computational and combinatorial property. In 2018 González A, Hernando C, and Mora M make a new definition related to the metric dimension and dominating set. They called the concept by Metric-locating-dominating sets. A dominating set of any graph is said to be a metric-locating-dominating set if the distance of each elements dominating set to another vertex of the graph is different. Another concept development of metric dimension was done by some researcher such as strong metric dimension by (Kuziak et al., 2013) and they can compute the metric dimension by the subgraph (Kuziak et al., 2017) and they continue for the next year to get the properties of strong resolving graphs (Kuziak et al., 2018). Next, strong metric dimension was modified as the fractional strong metric dimension (Kang et al., 2018).

Another concept that related to metric dimension is star metric dimension which was firstly proposed by Mutia, N., 2015. (Mutianingsih et al., 2016). They defined the star metric dimension and determine the star metric dimension of wheel-similar graph. The order set of $Z = \{z_1, z_2, \dots, z_n\} \subseteq V(G)$ called star resolving set of connected graph G if Z is Star Graph and for every vertex in G has different representation to the set Z. The representation is expressed as the distance $d(u,z)$, it is the shortest path from vertex u to z for every $u, z \in V(G)$. Star resolving set with the minimum cardinality is called star basis and its cardinality is called star metric dimension of G is $Sdim(G)$ (Mutia, 2015).

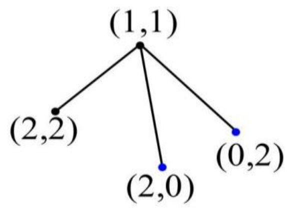

Figure 1. Star Graph (S_4) with $dim(S_4) = 2$

Figure 2 shows a wheel graph (W_6) with the vertex set is $V(W_6) = \{o, a, b, c, d, e, f\}$, let the star resolving set is $Z = \{o, a, b\}$. The representation every vertex of $V(W_6)$ to Z are:

$$r(o|Z) = \{0,1,1\}$$
$$r(a|Z) = \{1,1,1\}$$
$$r(b|Z) = \{1,1,0\}$$
$$r(c|Z) = \{1,2,1\}$$
$$r(d|Z) = \{1,2,2\}$$
$$r(e|Z) = \{1,1,2\}$$
$$r(f|Z) = \{1,0,2\}$$

We can see that every vertex of $V(W_6)$ has different representation to Z, then Z is star resolving set of W_6. Besides the Figure 3 show that if we take $|Z| = 2$, and suppose $Z = \{o, f\}$. Then, there are some vertices have same representation to Z, for example $r(a|Z) = \{1,1\} = r(e|Z) = \{1,1\}$. Thus $Z = \{o, f\}$ is not star resolving set of W_6. Therefore $|Z| = 3$ is star resolving set with minimum cardinality. The formal proof of star metric dimension of wheel graph has been found by (Mutia, 2015) as presented in Theorem 1. While Theorem 2 describe the star metric dimension of gear graph.

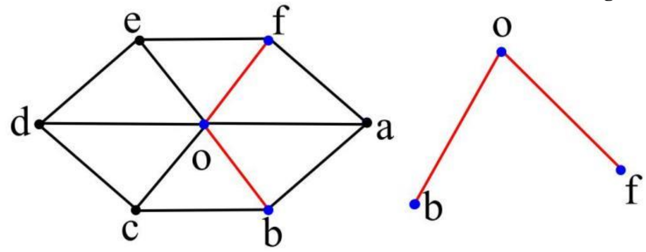

Figure 2. Wheel graph (W_6) with star resolving set equals three

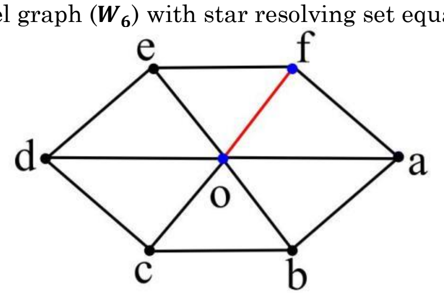

Figure 3. Wheel graph (W_6)

Theorem 1. If W_n is wheel graph with the order $n+1$ and $n \geq 7$, then star metric dimension $sdim(W_n) = \left\lceil \frac{n}{2} \right\rceil$. (Mutia, 2015)

Theorem 2. If $G_{2n}$is gir graph with the order order $2n+1$ and $n \geq 5$, then star metric dimension $sdim(G_{2n}) = n$. (Mutia, 2015)

Since the concept of star metric dimension is interesting to be developed because it still can be development with some classes of graph. The topic of star metric dimension is interesting. (Saifudin et al., 2021) succeed found the application of this topic to real life problem. They determined the aircraft navigation to protect a forest fire area by using the concept of star metric dimension. This topic still needed to be developed for any single graph or some products of graphs and some characteristic of star metric dimension not submitted yet. Therefore, in this paper we present some of characterization of star metric dimension and determine the star metric dimension of some special graph.

2. RESEARCH METHOD

The method which is used in this study are pattern recognition and axiomatic deductive methods. All the procedures are given below:

1) Reviewing the literature related to the concept of graph and star metric dimension.
2) Observing star metric dimension of some well know graphs.
3) Determining star metric dimension ($Sdim$) of graphs based on the procedure below:
   a) Determining the candidates of star local resolving set Z starting for the smallest cardinality
   b) If Z is Star Graph and for every vertex in G has different representation to the set Z. The representation is expressed as the distance $d(u,z)$, it is the shortest path from vertex u to z for every $u,z \in V(G)$.
   c) Verifying representation of each vertex u element G to local resolving set Z starting for the smallest cardinality.
   d) If the representation of each adjacent vertex to Z is different, then $Sdim(G) = |Z|$, if it is not, then repeat all steps until the biggest cardinality of Z.

3. RESULTS AND DISCUSSION

The following lemma and theorem shows the new results of star metric dimensions, starting with the characteristics of the star metric dimensions, then star metric dimensions on special graphs, i.e. complete, complete bipartite and fan graph:

Lemma 3. If G be a connected graph, $T \subseteq V(G)$, T consist of star resolving set of graph G and T form a star graph, then T is star metric dimension of G.

Proof. Let $Z = \{u_i | i = 1,2,\dots,k\} \subseteq V(G)$ is star resolving set of graph G and $T \subseteq V(G)$. Let $Z \subseteq T$, because Z star resolving set, for every $u,v \in V(G)$ then $r(u|Z) \neq r(v|Z)$ and $r(u|T) \neq r(v|T)$ because $Z \subseteq T$. Next, because T form star graph, then T is star resolving set of graph G.■

Lemma 4. If G be a connected graph, $Z \subseteq V(G)$ and For every $v_i, v_j \in Z$ then $r(v_i|Z) \neq r(v_j|Z)$.

Proof. Let $Z = \{v_1, v_2, v_3, \dots, v_k\} \subseteq V(G)$. Because for every $v_i, v_j \in Z$ with $i \neq j$ then $d(v_i, v_i) = 0$ and $d(v_i, v_j) \neq 0$, then there exist 0 of i^{th} element in $r(v_i|Z)$ for every $v_i \in Z$. As a result $r(v_i|W_l) \neq r(v_j|W_l)$ for $i \neq j$. ■

Theorem 5. If $n \geq 3$, then $Sdim(K_n) = n - 1$.

Proof. Let $V(K_n) = \{v_i | i = 1,2,3,\dots,n\}$ and $E(K_n) = \{v_i v_j | i,j = 1,2,3,\dots,n, i \neq j\}$ with $n \geq 3$. Choose $Z = \{v_i | i = 1,2,3,\dots,n - 1\}$ then $|Z| = n - 1$. See at **Lemma 4**, we get $r(v_i|Z) \neq r(v_j|Z)$ for every $v_i, v_j \in Z$ with $i \neq j$. Then, because $r(v_n|Z) = \left(\underbrace{1,1,1,1,\dots,1}_{(n-1)-tuple}\right)$, so W is resolving set of graph K_n. In other side, because for every $v_i \in Z$ with $i = 2,3,4,\dots,n-1$ then $v_1 v_i \in E(K_n)$, thus Z form star graph $K_{1,n-2}$ then Z is star resolving set of graph K_n. Next, because $dim\,(K_n) = n - 1$, it clearly that Z is star metric dimension with minimum cardinality then it can conclude that $Sdim(K_n) = n - 1$. ■

Theorem 6. If $n \geq 2$, then $Sdim\left(K_{2,n}\right) = n$.

Proof. Let $V\left(K_{2,n}\right) = \{a_1, a_2, b_i | i = 1,2,\dots,n\}$ and $E\left(K_{2,n}\right) = \left\{a_1 b_i, a_2 b_i \,\middle|\, i = 1,2,3,\dots,n\right\}$. Choose $Z = \{a_1, b_i | i = 1,2,3,\dots,n-1\}$ then $|Z| = n$. See at **Lemma 4**, we have $r\left(a_1\middle|Z\right) \neq r(b_i|Z)$ and $r(b_j|Z) \neq r(b_k|Z)$ for every $a_1, b_i, b_j, b_k \in Z$ with $j \neq k$. Then, because $r\left(a_2|Z\right) = \left(2,\underbrace{1,1,1,\dots,1}_{(n-1)-tuple}\right)$ and $r(b_n|Z) = \left(1,\underbrace{2,2,2,\dots,2}_{(n-1)-tuple}\right)$, thus Z is resolving set of $K_{2,n}$. In other side, because for every $b_i \in Z$ with $i = 1,2,3,\dots,n-1$ then $a_1 b_i \in E(K_{2,n})$, hence Z form star graph $K_{1,n-1}$ then Z is star resolving set of $K_{2,n}$. Next, because $dim\left(K_{2,n}\right) = n$, it clearly that Z is star resolving set with minimum cardinality. Therefore we can conclude that $Sdim\left(K_{2,n}\right) = n$. ■

Theorem 7. If $m,n \geq 3$, then $Sdim\left(K_{m,n}\right) = 0$.

Proof. Let $V(K_{m,n}) = \{a_i | i = 1,2,3,\dots,m\} \cup \{b_j | j = 1,2,\dots,n\}$ and $E(K_{m,n}) = \{a_i b_j | i = 1,2,3,\dots,m\,;\, j = 1,2,3,\dots,n\}$ with $m,n \geq 3$. Since $K_{m,n}$ is a complete bipartite graph, then there exist two possibilities for Z:

1. Select $Z = \{a_i | i = 2,3,4, \dots, m\} \cup \{b_j | j = 2,3,4, \dots, n\}$, so that $|Z| = m + n - 2$. Based on **Lemma 4**, we get that $r(u|Z) \neq r(v|Z)$ for every $u, v \in Z$ with $u \neq v$. Furthermore, Since $r(a_1|Z) = \left(\underbrace{2,2,2, \dots, 2}_{(m-1)-tuple}, \underbrace{1,1,1, \dots, 1}_{(n-1)-tuple}\right)$ and $r(b_1|Z) = \left(\underbrace{1,1,1, \dots, 1}_{(m-1)-tuple}, \underbrace{2,2,2, \dots, 2}_{(n-1)-tuple}\right)$, then W is resolving set of $K_{m,n}$. Furthermore, Since Z doesn't form a star, then Z is not a star resolving set of $K_{m,n}$.
2. Select $Z = \{a_1, b_1\}$ so that $|Z| = 2$. Since $dim\,(K_{m,n}) = m + n - 2$, it clearly shows that Z is not a resolving set of $K_{m,n}$ so that Z is not a star resolving set of $K_{m,n}$.

Furthermore, Since $dim\,(K_{m,n}) = m + n - 2$, then $Sdim\,(K_{m,n}) \geq m + n - 2$. On the other hand, Since star subgraph of $K_{m,n}$ only can be obtained by $m + 1$ or $n + 1$ vertices, then $Sdim\,(K_{m,n}) \leq m + 1$ or $Sdim\,(K_{m,n}) \leq n + 1$. Contradiction with the fact that $Sdim\,(K_{m,n}) \geq m + n - 2$. Consequently, there is no vertex set as star resolving set of $K_{m,n}$ so that $Sdim\,(K_{m,n}) = 0$. ■

Theorem 8. If $2 \leq n \leq 5$, then $Sdim\,(F_n) = 2$.

Proof. Let $V(F_n) = \{v, v_i | i = 1,2, \dots, n\}$ and $E(F_n) = \{vv_i | i = 1,2, \dots, n\} \cup \{v_j v_{j+1} | j = 1,2, \dots, n-1\}$. There are for cases for n:

1. For $n = 2$, select $Z = \{v, v_1\}$ so that $|Z| = 2$. Since $r(v|Z) = (0,1)$, $r(v_1|Z) = (1,0)$ and $r(v_2|Z) = (1,1)$, then Z is a resolving set of F_2. While, since Z forms $K_{1,1}$ star, then Z is a star resolving set of F_2. Next, take any singleton. Let $T = \{v\}$, then we get $r(v_1|T) = r(v_2|T) = (1,1)$ so that T is not a resolving set of F_2. Consequently, T is not a resolving set of F_2. So that, Z is star resolving set with minimum cardinality and $Sdim(F_2) = 2$.
2. For $n = 3$, select $Z = \{v_1, v_2\}$ so that $|Z| = 2$. Since $r(v|Z) = (1,1)$, $r(v_1|Z) = (0,1)$, $r(v_2|Z) = (1,0)$ and $r(v_3|Z) = (2,1)$, then Z is a resolving set of F_3. While, since Z forms a $K_{1,1}$, then Z is star resolving set of F_3. Next, take any singleton. There are two possibilities: (i) v with $\deg(v) = 2$; or (ii) v with $\deg(v) = 3$.
   (i) Let $T = \{v_1\}$, then we get $r(v|T) = r(v_2|T) = (1)$ so that T is not a resolving set of F_3. Consequently, T is not a resolving set of F_3.
   (ii) Let $T = \{v\}$, then we get $r(v_1|T) = r(v_2|T) = r(v_3|T) = (1)$ so that T is not a resolving set of F_3. Consequently, T is not a resolving set of F_3.

   So that, Z is star resolving set with minimum cardinality and $Sdim(F_3) = 2$.
3. For $n = 4$, select $Z = \{v_2, v_3\}$ so that $|Z| = 2$. Since $r(v|Z) = (1,1)$, $r(v_1|Z) = (1,2)$, $r(v_2|Z) = (0,1)$, $r(v_3|Z) = (1,0)$ and $r(v_4|Z) = (2,1)$, then Z is a resolving set of F_4. While, since Z forms a $K_{1,1}$, then Z is star resolving set of F_4. Next, take any singleton. There are two possibilities: (i) v with $\deg(v) = 2$; or (ii) v with $\deg(v) = 3$.
   (i) Let $T = \{v_1\}$, then we get $r(v|T) = r(v_2|T) = (1)$ so that T is not a resolving set of F_4. Consequently, T is not a resolving set of F_4.

(ii) Let $T = \{v\}$, then we get $r\left(v_1|T\right) = r\left(v_2|T\right) = r\left(v_3|T\right) = r\left(v_4|T\right) = (1)$ so that T is not a resolving set of F_4. Consequently, T is not a resolving set of F_3.

So that, Z is a star resolving set with minimum cardinality and $Sdim\left(F_4\right) = 2$.

4. For $n = 5$, select $Z = \{v_2, v_3\}$ so that $|Z| = 2$. Since $r(v|Z) = (1,1)$, $r\left(v_1|Z\right) = (1,2)$, $r\left(v_2|Z\right) = (0,1)$, $r\left(v_3|Z\right) = (1,0)$, $r\left(v_4|Z\right) = (2,1)$ and $r\left(v_5|Z\right) = (2,2)$, then Z is a resolving set of F_5. On the other hand, Since Z forms a $K_{1,1}$, then Z is a star resolving set of F_5. Furthermore, take any singleton. There are two possibilities: (i) v with $\deg(v) = 2$; or (ii) v with $\deg(v) = 3$.

(i) Let $T = \{v_1\}$, then we get $r(v|T) = r\left(v_2|T\right) = (1)$ so that T is not a resolving set of F_5. Consequently, T is not a resolving set of F_5.

(ii) Let $T = \{v\}$, then we get $r\left(v_1|T\right) = r\left(v_2|T\right) = r\left(v_3|T\right) = r\left(v_4|T\right) = r\left(v_5|T\right) = (1)$ so that T is not a resolving set of F_5. Consequently, T is not a resolving set of F_5.

So that, Z is a star resolving set with minimum cardinality and $Sdim\left(F_5\right) = 2$.

From the four possibilities above, we get that for $n = 2,3,4,5$, $Sdim(F_n) = 2$. ■

Theorem 9. If $n \geq 6$, then $Sdim\ (F_n) = \left\lceil\frac{2n+3}{5}\right\rceil$.

Proof. Let $V(F_n) = \{v, v_i | i = 1,2, \dots, n\}$ and $E(F_n) = \{vv_i | i = 1,2, \dots, n\} \cup \{v_j v_{j+1} | j = 1,2, \dots, n-1\}$. There are three cases for n:

1. For $n \equiv 1\ (mod\ 5)$ or $n \equiv 3\ (mod\ 5)$, select $Z = \{v, v_2, v_4, v_7, v_9, \dots, v_{5i-3}, v_{5i-1}, \dots, v_{n-2}\}$ so that $|Z| = \left\lceil\frac{2n+3}{5}\right\rceil$. Based on **Lemma 4**, we get that $r(u|Z) \neq r(v|Z)$ for every $u, v \in Z$ with $u \neq v$. Furthermore, Since for every $v_i \in V(F_n)$ we get:

$$r(v_i|Z) = \begin{cases} \left(1,2,2,2,\dots,2, \underbrace{1}_{\left(\frac{i}{3}+1\right)^{th}}, 2,2,2,\dots,2\right), & i \equiv 0\ (mod\ 5) \\ \left(1,2,2,2,\dots,2, \underbrace{1}_{\left(\left\lceil\frac{i}{2}\right\rceil - \left\lfloor\frac{i}{10}\right\rfloor + 1\right)^{th}}, 2,2,2,\dots,2\right), & i \equiv 1\ (mod\ 5) \\ \left(1,2,2,2,\dots,2, \underbrace{1}_{\left(\left\lceil\frac{i}{3}\right\rceil + \left\lfloor\frac{i}{15}\right\rfloor + 1\right)^{th}}, \underbrace{1}_{\left(\left\lceil\frac{i}{3}\right\rceil + \left\lfloor\frac{i}{15}\right\rfloor + 2\right)^{th}}, 2,2,2,\dots,2\right), & i \equiv 3\ (mod\ 5) \end{cases}$$

So that, Z is a resolving set of F_n. Furthermore, Since for every $v_{5i-3}, v_{5i-1} \in Z$ we get $vv_{5i-3} \in E(F_n)$ and $vv_{5i-1} \in E(F_n)$, then Z forms a $K_{1,\left\lceil\frac{2n+3}{5}\right\rceil - 1}$ so that Z is a star resolving set of F_n. Furthermore, take any $T \subseteq V(F_n)$ with $|T| < |Z|$, $|T| = |Z| - 1$. Then there are two cases for T:

(i) T doesn't consist v. Let $T = \{v_2, v_4, v_7, v_9, \dots, v_{5i-3}, v_{5i-1}, \dots, v_{n-2}\}$. Then T doesn't form a star.

(ii) T consists v. Let $T = \{v, v_2, v_4, v_7, v_9, \dots, v_{5i-3}, v_{5i-1}, \dots, v_{n-7}\}$. There exist $v_{n-2}, v_{n-1}, v_n \in V(F_n) - T$, so that $r(v_{n-2}|T) = r(v_{n-1}|T) = r(v_n|T) = (2,2,2,\dots,2)$. Consequently, T is not a resolving set of F_n.

So that, there is no star resolving set with cardinality less then $|Z|$ so that Z is a star resolving set with minimum cardinality and $Sdim\ (F_n) = \left\lceil\frac{2n+3}{5}\right\rceil$ For $n \equiv 1\ (mod\ 5)$ or $n \equiv 3\ (mod\ 5)$.

2. For $n \equiv 2 \ (mod\ 5)$ or $n \equiv 4 \ (mod\ 5)$, select $Z = \{v, v_2, v_4, v_7, v_9, \dots, v_{5i-3}, v_{5i-1}, \dots, v_n\}$ so that $|Z| = \left\lceil \frac{2n+3}{5} \right\rceil$. Based on **Lemma 4**, we get that $r(u|Z) \neq r(v|Z)$ for every $u, v \in Z$ with $u \neq v$. Furthermore, Since for every $v_i \in V(F_n)$ we get:

$$r(v_i|Z) = \begin{cases} \left(1,2,2,2,\dots,2, \underbrace{1}_{\left(\frac{i}{3}+1\right)^{th}}, 2,2,2,\dots,2\right), & i \equiv 0 \ (mod\ 5) \\ \left(1,2,2,2,\dots,2, \underbrace{1}_{\left(\left\lceil\frac{i}{2}\right\rceil - \left\lfloor\frac{i}{10}\right\rfloor+1\right)^{th}}, 2,2,2,\dots,2\right), & i \equiv 1 \ (mod\ 5) \\ \left(1,2,2,2,\dots,2, \underbrace{1}_{\left(\left\lceil\frac{i}{3}\right\rceil + \left\lfloor\frac{i}{15}\right\rfloor+1\right)^{th}}, \underbrace{1}_{\left(\left\lceil\frac{i}{3}\right\rceil + \left\lfloor\frac{i}{15}\right\rfloor+2\right)^{th}}, 2,2,2,\dots,2\right), & i \equiv 3 \ (mod\ 5) \end{cases}$$

So that, Z is a resolving set of F_n. Furthermore, Since for every $v_{5i-3}, v_{5i-1} \in Z$ we get $vv_{5i-3} \in E(F_n)$ and $vv_{5i-1} \in E(F_n)$, then Z forms a $K_{1,\left\lceil\frac{2n+3}{5}\right\rceil-1}$ so that Z is a star resolving set of F_n. Furthermore, take any $T \subseteq V(F_n)$ with $|T| < |Z|$, $|T| = |Z| - 1$. Then there are two cases for T:

(i) T doesn't consist v. Let $T = \{v_2, v_4, v_7, v_9, \dots, v_{5i-3}, v_{5i-1}, \dots, v_n\}$. Then T doesn't form a star.

(ii) T consists v. Let $T = \{v, v_2, v_4, v_7, v_9, \dots, v_{5i-3}, v_{5i-1}, \dots, v_{n-5}\}$. There exists $v_{n-2}, v_{n-1}, v_n \in V(F_n) - T$, so that $r(v_{n-2}|T) = r(v_{n-1}|T) = r(v_n|T) = (2,2,2,\dots,2)$. Consequently, T is not a resolving set of F_n.

So that, there is no star resolving set with cardinality less then $|Z|$ so that Z is a star resolving set with minimum cardinality and $Sdim\ (F_n) = \left\lceil\frac{2n+3}{5}\right\rceil$ For $n \equiv 2 \ (mod\ 5)$ or $n \equiv 4 \ (mod\ 5)$.

3. For $n \equiv 0 \ (mod\ 5)$, select $Z = \{v, v_2, v_4, v_7, v_9, \dots, v_{5i-3}, v_{5i-1}, \dots, v_{n-1}\}$ so that $|Z| = \left\lceil\frac{2n+3}{5}\right\rceil$. Based on **Lemma 4**, we get that $r(u|Z) \neq r(v|Z)$ for every $u, v \in Z$ with $u \neq v$. Furthermore, Since for every $v_i \in V(F_n)$ we get:

$$r(v_i|Z) = \begin{cases} \left(1,2,2,2,\dots,2, \underbrace{1}_{\left(\frac{i}{3}+1\right)^{th}}, 2,2,2,\dots,2\right), & i \equiv 0 \ (mod\ 5) \\ \left(1,2,2,2,\dots,2, \underbrace{1}_{\left(\left\lceil\frac{i}{2}\right\rceil - \left\lfloor\frac{i}{10}\right\rfloor+1\right)^{th}}, 2,2,2,\dots,2\right), & i \equiv 1 \ (mod\ 5) \\ \left(1,2,2,2,\dots,2, \underbrace{1}_{\left(\left\lceil\frac{i}{3}\right\rceil + \left\lfloor\frac{i}{15}\right\rfloor+1\right)^{th}}, \underbrace{1}_{\left(\left\lceil\frac{i}{3}\right\rceil + \left\lfloor\frac{i}{15}\right\rfloor+2\right)^{th}}, 2,2,2,\dots,2\right), & i \equiv 3 \ (mod\ 5) \end{cases}$$

So that, Z is a resolving set of F_n. Furthermore, Since for every $v_{5i-3}, v_{5i-1} \in Z$ we get $vv_{5i-3} \in E(F_n)$ and $vv_{5i-1} \in E(F_n)$, then Z forms a $K_{1,\left\lceil\frac{2n+3}{5}\right\rceil-1}$ so that Z is a star resolving set of F_n. Furthermore, take any $T \subseteq V(F_n)$ with $|T| < |Z|$, $|T| = |Z| - 1$. Then there are two cases for T:

(iii) T doesn't consist v. Let $T = \{v_2, v_4, v_7, v_9, \dots, v_{5i-3}, v_{5i-1}, \dots, v_{n-1}\}$. Then T doesn't form a star.

(iv) T consist v. Let $T = \{v, v_2, v_4, v_7, v_9, \dots, v_{5i-3}, v_{5i-1}, \dots, v_{n-6}\}$. There exists $v_{n-2}, v_{n-1}, v_n \in V(F_n) - T$, so that $r(v_{n-2}|T) = r(v_{n-1}|T) = r(v_n|T) = (2,2,2,\dots,2)$. Consequently, T is not a resolving set of F_n.

So that, there is no star resolving set with cardinality less then $|Z|$ so that Z is a star resolving set with minimum cardinality and $Sdim\ (F_n) = \left\lceil\frac{2n+3}{5}\right\rceil$ For $n \equiv 0 \ (mod\ 5)$ or $n \equiv 3 \ (mod\ 5)$.

Based on three cases above, we get that for $n \geq 6$ we get $Sdim(F_n) = \left\lceil \frac{2n+3}{5} \right\rceil$. ■

4. CONCLUSION

Based on the all the explanations above, we can conclude that complete graph has $\boldsymbol{Sdim(K_n) = n - 1}$, for $\boldsymbol{n \geq 3}$, bipartite graph $\boldsymbol{K_{2,n}}$ has $\boldsymbol{Sdim\left(K_{2,n}\right) = n}$, for $\boldsymbol{n \geq 3}$. Besides complete bipartite graph hasn't star metric dimension or for $\boldsymbol{m, n \geq 3}$ or it can said that $\boldsymbol{Sdim\left(K_{m,n}\right) = 0}$. Another graph, that is Fan graph has $\boldsymbol{Sdim\,(F_n) = 2}$ for $\boldsymbol{2 \leq n \leq 5}$ and for $\boldsymbol{\geq 6}$ $\boldsymbol{Sdim\,(F_n) = \left\lceil \frac{2n+3}{5} \right\rceil}$. We also give some an issue regarding the topic of star metric dimension for another researcher as follows. If given a connected graph $\boldsymbol{G}$ with the order equals $\boldsymbol{n}$ vertex, determine star metric dimension of some product of graphs.

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AUTHOR'S CONTRIBUTIONS

The authors discussed the results and contributed to from the start to final manuscript.

CONFLICT OF INTEREST

There are no conflicts of interest declared by the authors.

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